Exact algebraic separability criterion for two-qubit systems†

K. Fujikawa*1 and C. H. Oh*2

A conceptually simpler proof of the separability criterion for two-qubit systems, which is referred to as "Hefei inequality" in literature,[1] is analyzed. This inequality gives a necessary and sufficient separability criterion for any mixed two-qubit system unlike the Bell-CHSH inequality[2,3] which cannot test mixed-states such as the Werner state[4] when regarded as a separability criterion. The original derivation of this inequality[1] emphasized the uncertainty relation of complementary observables; however, we show that the uncertainty relation does not play any role in the actual derivation and that the Peres-Hodrodecki condition[5] is solely responsible for the inequality. Our derivation, which contains technically novel aspects such as an analogy to the Dirac equation, sheds light on this inequality and on the fundamental issue of the to extent to which the uncertainty relation can provide a test of entanglement. This separability criterion is illustrated for an exact treatment of the Werner state.

Our starting point is the fact that the general *pure* two-qubit states are brought to the standard form by the Schmidt decomposition

$$|\Phi\rangle = (u \otimes v)[s_1|+\rangle \otimes |-\rangle - s_2 e^{i\delta}|-\rangle \otimes |+\rangle] \qquad (1)$$

with

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (2)$$

and real numbers $s_1^2 + s_2^2 = 1$ and δ. Namely, the states are parametrized by s_1, s_2, δ and two unitary matrices u and v. It is then shown that this system is represented formally in terms of a 4-dimensional Dirac notation.

We then obtain the inequalities (*separability criterion*)

$$\begin{aligned} \langle P_-\rangle_\rho^2 &\geq \langle \gamma_3\gamma_0 P_-\rangle_\rho^2 + \langle \gamma_2\gamma_0 P_-\rangle_\rho^2 + \langle \gamma_1\gamma_0 P_-\rangle_\rho^2, \\ \langle P_+\rangle_\rho^2 &\geq \langle \gamma_3\gamma_0 P_+\rangle_\rho^2 + \langle \gamma_2\gamma_0 P_-\rangle_\rho^2 + \langle \gamma_1\gamma_0 P_-\rangle_\rho^2, \end{aligned} \qquad (3)$$

where $P_\pm = (1 \pm \gamma_5)/2$ and

$$\langle \gamma_3\gamma_0 P_\pm\rangle_\rho \equiv \mathrm{Tr}\gamma_3\gamma_0 P_\pm \rho, \qquad (4)$$

for example, using the Peres-Horodecki criterion without referring to the uncertainty relations.

As for the test of the Werner state[4] which is defined by

$$\rho_w = \frac{1}{4}(1-\beta)\mathbf{1} + \beta|\psi_s\rangle\langle\psi_s| \qquad (5)$$

with the singlet state $|\psi_s\rangle = (1/\sqrt{2})[|+\rangle|-\rangle - |-\rangle|+\rangle]$, we obtain

$$1 \geq \beta + 2\beta. \qquad (6)$$

We thus conclude that the separability condition of the Werner state is equivalent to

$$\beta \leq \frac{1}{3}, \qquad (7)$$

which agrees with the result of a more explicit analysis of ρ_w.[4] This in particular implies that $\beta > \frac{1}{3}$ stands for an *inseparable state*.

We have re-analyzed one of the representative inequalities proposed in Ref. 1) and have shown that the uncertainty relations cannot be alternative to the Peres-Horodecki condition in the analysis of entanglement for general two-qubit systems. The "Hefei inequality," however, stands for a rare algebraic criterion that is applicable to any mixed state that cannot be tested by the Bell-CHSH inequality in general,[2,3] as was illustrated by an exact treatment of the Werner state.

Here, in comparison with the criterions of separability of two-qubit systems, we briefly mention a corresponding test of the separability of systems with two continuous degrees of freedom.[6–8] In the problem of two-party continuum case with two-dimensional continuous phase space freedom (p, q) in each party, it is possible to re-formulate the problem such that[8]

(1) the uncertainty relation leads to a necessary condition for separable two-party systems,
(2) the derived condition is sufficient to prove the separability of two-party Gaussian systems.

Namely, the uncertainty relation *without* referring to the Peres-Horodecki criterion[5] provides a necessary and sufficient separability condition for two-party Gaussian systems.

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*1 RIKEN Nishina Center
*2 Department of Physics, National University of Singapore

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